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Appell-Hamel dynamical system: a nonlinear test of the Chetaev and the vakonomic model

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We contrast the results of the Chetaev and vakonomic models of nonholonomic mechanics when applied to the non-holonomically constrainted Appell-Hamel dynamical system with a nonvanishing constant force parallel to the tangent direction of the wheel. We find that the Chetaev result is compatible with Newton-Euler solution while the vakonomic model yields results inconsistent with the predictions of the Newton-Euler mechanics.

1 Introduction

Numerous problems in wheeled dynamics, robotics, and motion generation requires the use of nonholonomic mechanics (cf. [1, 2] and references therein). In contrast to *holonomically* constrained problems where the dependent coordinates are eliminated by exploiting the constraint equation to yield the independent coordinates, the generalized coordinates [3], for *noholonomic* constraints dependent coordinates are not deducible from the constraint equations and only some dependent velocities can be solved for. Hence, standard methods of holonomic and nonholonomic mechanics are qualitatively different. Two basics models that have been put forward in the realm of analytical nonholonomic mechanics are the Chetaev model [4–7] and vakonomic model [8–12]. The former has been proposed first by Chetaev and consists in obtaining the virtual displacement condition by multiplying the virtual displacements with the velocity gradients of constraint equation. However, this doing has not yet been justified starting from the basic principles of mechanics. The vakonomic model has been developed by Kozlov in the early 1980's. In this case the virtual displacement condition follows from a straightforward variation of the constraint equation. The differential equation thus obtained is called "the geodesic trajectory equation". Lewis et al. [13] showed experimentally that such equation yields predictions inconsistent with the real dynamics for some systems.

As a test for the Chetaev and vakonomic models we investigate in this work the Appell-Hamel dynamical system [14–16] in the frame works of Newton-Euler mechanics and analytical mechanics. This problem is one of the classical example of nonlinear nonholonomic system [17–19]. Upon applying a nonvanishing constant force along the tangent direction of the wheel trace we inspect the consistency of the results of the Chetaev and vakonomic models with the real motion as described by Newton-Euler solutions. The organization is as follows: In Sect. 2, we introduce the Appell-Hamel dynamical system and derives its Newton-Euler solution. In Sect. 3, we discuss the Chetaev solution and compare it with the Newton-Euler result. In Sect. 4, we substitute the Newton-Euler solution into the "the geodesic trajectory equation" of the vakonomic model and find inconsistency. In Sect. 5 we summarize and conclude.

2 The Newton-Euler solution

The Appell-Hamel problem, schematically depicted in Fig. 1, consists of a block of mass m that is confined vertically and is attached to the end of a thread that passes over two small massless pulleys and is wound around a drum of radius b. The

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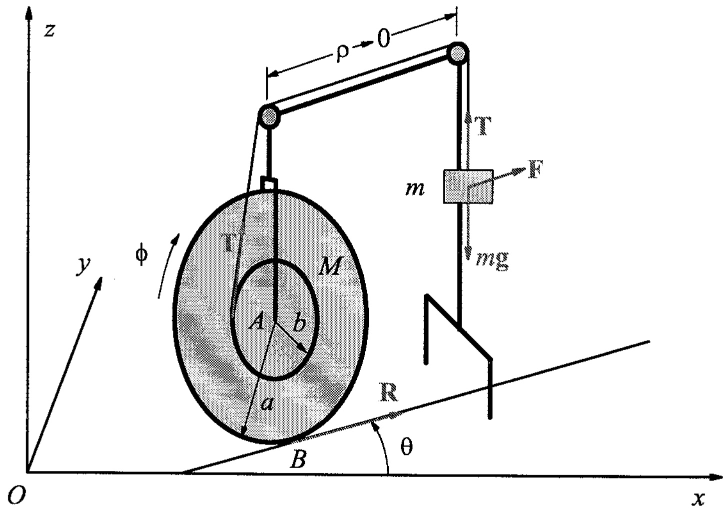

Fig. 1 The Appel-Hamel dynamical system with a nonvanishing constant force **F** applied parallel to the tangent direction of the wheel trace.

latter is rigidly attached to a wheel of radius a which rolls on a horizontal xy-plane. The mass of the wheel is M, and the moment of inertia of the wheel and drum around the axis across A is I. The legs of the frame supporting the pulleys and keeping the wheel vertical slide on the xy-plane without friction. The moment of inertia of the system around the vertical wheel diameter is I'. In the present work we assume in addition that a nonvanishing constant force **F** is applied parallel to the tangent direction of the wheel trace (cf. Fig. 1). Such a force may be realized, for instance, by a rocket oriented in the tangent direction. Let x, y, z be the coordinates of the mass m, and θ the angle between the plane of the wheel and the x-axis. The angle ϕ describes the rotation of the wheel in its own plane. The distance between the centers of both pulleys is ρ. For simplicity, we let $\rho \to 0$. The nonholonomic constraints are given by

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0, \tag{1}$$

$$\dot{x}\cos\theta + \dot{y}\sin\theta = a\dot{\phi}, \tag{2}$$

$$\dot{z} = -b\dot{\phi}. \tag{3}$$

From these equations we deduce a constraint equation that is nonlinear in the velocities, i.e.

$$\dot{x}^2 + \dot{y}^2 = \frac{a^2}{b^2}\dot{z}^2, \tag{4}$$

or

$$\dot{z} = -\frac{b}{a}\sqrt{\dot{x}^2 + \dot{y}^2}. \tag{5}$$

At first, we solve the Appell-Hamel problem in the framework of Newton-Euler mechanics. Suppose the tension in the rope is **T**, the friction in the contact point B is **R**, and the torque along the vertical symmetry axis vanishes. Newton-Euler equations read then

$$F + R = (M + m)A_\tau, \tag{6}$$

$$T - mg = m\ddot{z}, \tag{7}$$

$$Tb - Ra = I\ddot{\phi}, \tag{8}$$

$$I'\ddot{\theta} = 0. \tag{9}$$

Among these equations, Eq. (6) describes the horizontal motion of the dynamical system as a whole, where A_τ is the tangent component of the acceleration of the mass; Eq. (7) describes the vertical motion of the block; Eq. (8) describes the rotation of the wheel and drum around their common axis A; Eq. (9) describes the rotation of the system around the vertical diameter of the wheel. According to Eqs. (1) and (2),

$$A_\tau = \frac{d}{dt}(\dot{x}\cos\theta + \dot{y}\sin\theta) = \frac{d}{dt}\sqrt{\dot{x}^2 + \dot{y}^2} = a\ddot{\phi}. \tag{10}$$

Solving Eqs. (6)–(10), we obtain

$$\ddot{\phi} = \frac{Fa + mgb}{I + (m+M)a^2 + mb^2}. \tag{11}$$

Therefore,

$$A_\tau = \frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a. \tag{12}$$

Let us inspect the solutions for the following two special initial conditions:

i) The system starts at (x_0, y_0) from still, and the wheel does not rotate along its vertical symmetry axis and is initially orientated in the θ_0 direction, meaning that

$$x(0) = x_0,\ y(0) = y_0, \tag{13}$$

$$\dot{x}(0) = \dot{y}(0) = 0, \tag{14}$$

$$\theta(0) = \theta_0, \tag{15}$$

$$\dot{\theta}(0) = 0. \tag{16}$$

In this case the trace of wheel is a straight line across (x_0, y_0)

$$x = x_0 + \frac{1}{2}\frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a t^2 \cos\theta_0, \tag{17}$$

$$y = y_0 + \frac{1}{2}\frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a t^2 \sin\theta_0, \tag{18}$$

$$\theta = \theta_0. \tag{19}$$

ii) The system starts at the origin and the wheel rotates around its vertical symmetry with a constant angular frequency ω, i.e.

$$x(0) = y(0) = 0, \tag{20}$$

$$\dot{x}(0) = \dot{y}(0) = 0, \tag{21}$$

$$\theta(0) = \theta_0, \tag{22}$$

$$\dot{\theta}(0) = \omega. \tag{23}$$

For this situation we have

$$\dot{x} = \frac{1}{2}\frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a t^2 \cos\omega t, \tag{24}$$

$$\dot{y} = \frac{1}{2}\frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a t^2 \sin\omega t, \tag{25}$$

$$\dot{\theta} = \omega. \tag{26}$$

Integrating once more we finally obtain the solution:

$$x = \frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a \left(\frac{t}{\omega} \sin\omega t + \frac{1}{\omega^2} \cos\omega t - \frac{1}{\omega^2} \right), \tag{27}$$

$$y = \frac{Fa + mgb}{I + (m+M)a^2 + mb^2} a \left(-\frac{t}{\omega} \cos\omega t + \frac{1}{\omega^2} \sin\omega t \right), \tag{28}$$

$$\theta = \omega t. \tag{29}$$

The wheel trace represented by Eqs. (27), (28) is depicted in Fig. 2. The normal force for the curved motion is supplied by the constraint.

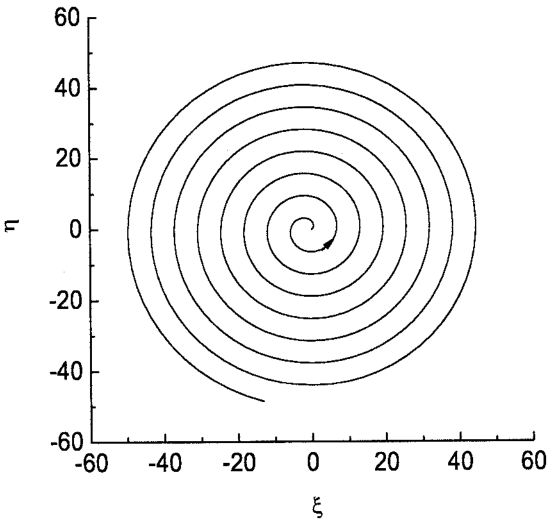

Fig. 2 The wheel trace of the Appel-Hamel dynamical system. Where $\xi = (\omega^2/A_\tau)x$ and $\eta = (\omega^2/A_\tau)y$ are dimensionless variables, with $A_\tau = (Fa+mgb)a/[I+(m+M)a^2+mb^2]$ being the tangent acceleration [see Eq. (12)].

3 The Chetaev solution

Now we discuss the Appell-Hamel problem following the Chetaev approach.

Suppose a system described by the Lagrangian $L(q,\dot{q},t)$ (where q represent the generalized coordinates $q_1,\cdots,q_n$) is subjected to the following nonholonomic constraints:

$$f_j(q,\dot{q},t)=0,\ (j=1,\cdots,k). \tag{30}$$

In the Chetaev model, the virtual displacement equation is given by the Chetaev condition:

$$\sum_{\alpha=1}^{n}\frac{\partial f_j}{\partial \dot{q}_\alpha}\delta q_\alpha=0,\ (j=1,\cdots,k). \tag{31}$$

Based on this relation the equation of motion is obtained as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=Q_\alpha+\sum_{j=1}^{k}\lambda_j\frac{\partial f_j}{\partial \dot{q}_\alpha},\ (\alpha=1,\cdots,n). \tag{32}$$

Where $Q_1,\cdots,Q_n$ are the nonconservative forces. $\lambda_1,\cdots,\lambda_k$ are undetermined multipliers, and $\lambda_j\partial f_j/\partial\dot{q}_\alpha$ represents the αth component of the constraint force of the jth constraint [20].

For simplicity, we assume that in the Appell-Hamel dynamical system all the mass of the wheel and drum is gathered at the center A, thus $I=0$. The Lagrangian of the system reads,

$$L=\frac{1}{2}(m+M)(\dot{x}^2+\dot{y}^2)+\frac{1}{2}m\dot{z}^2+\frac{1}{2}I'\dot{\theta}^2-mgz, \tag{33}$$

and the generalized forces are

$$Q_x=F\cos\theta,\ Q_y=F\sin\theta,\ Q_\theta=0. \tag{34}$$

Substituting the Lagrangian (33) and the constraints (1) and (4) into Eq. (32), we obtain

$$(m+M)\ddot{x}=F\cos\theta+\lambda_1\sin\theta+\lambda_2\dot{x}, \tag{35}$$

$$(m+M)\ddot{y}=F\sin\theta-\lambda_1\cos\theta+\lambda_2\dot{y}, \tag{36}$$

$$m\ddot{z}=-mg-\lambda_2\frac{a^2}{b^2}\dot{z}, \tag{37}$$

$$I'\ddot{\theta}=0. \tag{38}$$

Now we solve for the acceleration tangent to the wheel trace from the equations of motion (35)–(37). Differentiating the constraint equation (4) with respect to time yields

$$\ddot{z}=\frac{b^2}{a^2\dot{z}}(\dot{x}\ddot{x}+\dot{y}\ddot{y})\ . \tag{39}$$

Inserting (39) and (5) into (37) we conclude that

$$-m\frac{b}{a}\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -mg+\lambda_2\frac{a}{b}\sqrt{\dot{x}^2+\dot{y}^2}. \tag{40}$$

This relation leads to

$$\lambda_2 = -m\frac{b^2}{a^2}\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\dot{x}^2+\dot{y}^2}+mg\frac{b}{a}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}. \tag{41}$$

Inserting (41) into Eqs. (35) and (36) yields

$$(m+M)\ddot{x}+m\frac{b^2}{a^2}\frac{\dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{\dot{x}^2+\dot{y}^2} = mg\frac{b}{a}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}+F\cos\theta+\lambda_1\sin\theta, \tag{42}$$

$$(m+M)\ddot{y}+m\frac{b^2}{a^2}\frac{\dot{y}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{\dot{x}^2+\dot{y}^2} = mg\frac{b}{a}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}+F\sin\theta-\lambda_1\cos\theta. \tag{43}$$

Upon using the constraint (1) and the relation $\dot{x}\cos\theta+\dot{y}\sin\theta=\sqrt{\dot{x}^2+\dot{y}^2}$. In Eqs. (42) and (43), we deduce that

$$\dot{x}\ddot{x}+\dot{y}\ddot{y} = \frac{mgb+Fa}{mb^2+(m+M)a^2}a\sqrt{\dot{x}^2+\dot{y}^2}, \tag{44}$$

i.e.,

$$A_\tau = \frac{d}{dt}\sqrt{\dot{x}^2+\dot{y}^2} = \frac{mgb+Fa}{mb^2+(m+M)a^2}a. \tag{45}$$

The Chetaev solution (45) conforms with the Newton-Euler solution (12) for the case $I=0$. This confirms the consistency of the Chetaev model with the Newton-Euler solution, and we conclude thus that the Chetaev approach is applicable to the treatment of Appell-Hamel dynamical systems.

4 The vakonomic model

Now we show the failure of the vakonomic model in obtaining the correct solution of the Appell-Hamel problem. Since integrating the vakonomic equation directly is difficult, we change to another approach: first we insert the Newton-Euler solution into the "geodesic trajectory equation", and then find out inconsistency.

In the vakonomic model, the virtual displacement condition for a general nonholonomic constraint (30) is obtained by variating it with respect to the generalized coordinates and the generalized velocities

$$\sum_{\alpha=1}^{n}\left(\frac{\partial f_j}{\partial q_\alpha}\delta q_\alpha+\frac{\partial f_j}{\partial\dot{q}_\alpha}\delta\dot{q}_\alpha\right)=0,\ (j=1,\cdots,k), \tag{46}$$

and the "geodesic trajectory equation" reads [8–12]

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=Q_\alpha+\sum_{j=1}^{k}\lambda_j\left(\frac{\partial f_j}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial f_j}{\partial\dot{q}_\alpha}\right)-\sum_{j=1}^{k}\dot{\lambda}_j\frac{\partial f_j}{\partial\dot{q}_\alpha},\ (\alpha=1,\cdots,n). \tag{47}$$

For the Appell-Hamel system with $I=0$, Eq. (47) yields

$$(m+M)\ddot{x}=F\cos\theta+\frac{d}{dt}(\lambda_1\sin\theta+2\lambda_2\dot{x}), \tag{48}$$

$$(m+M)\ddot{y}=F\sin\theta-\frac{d}{dt}(\lambda_1\cos\theta+2\lambda_2\dot{y}), \tag{49}$$

$$m\ddot{z}=-mg+\frac{a^2}{b^2}\frac{d}{dt}(2\lambda_2\dot{z}), \tag{50}$$

$$I'\ddot{\theta}=\lambda_1(\dot{x}\cos\theta+\dot{y}\sin\theta). \tag{51}$$

Inserting the Newton-Euler solution (24)–(26) into Eqs. (48), (49), and (51) yields

$$\lambda_1=0 \tag{52}$$

$$(m+M)A_\tau(\cos\omega t - \omega t\sin\omega t) = F\cos\omega t - A_\tau\frac{d}{dt}(2\lambda_2 t\cos\omega t), \tag{53}$$

$$(m+M)A_\tau(\sin\omega t + \omega t\cos\omega t) = F\sin\omega t - A_\tau\frac{d}{dt}(2\lambda_2 t\sin\omega t). \tag{54}$$

Where A_τ is the constant acceleration given by (45). From Eqs. (53) and (54) we obtain further

$$(m+M)A_\tau = F - A_\tau\frac{d}{dt}(2\lambda_2 t), \tag{55}$$

$$-(m+M) = 2\lambda_2. \tag{56}$$

Substituting (56) into (55), yields,

$$F = 0! \tag{57}$$

This result is obviously in contradiction with our original presupposition. Therefore the vakonomic model does not conform with Newton-Euler mechanics and thus does not describe the actual motion of the Appell-Hamel dynamical system. In fact, as we have shown recently [21], in such a model, the virtual work of the reaction forces is nonvanishing and is unjustifiably neglected. In this sense, the constraint in the vakonomic model is not ideal.

5 Conclusions and final remarks

In this work we derived the Newton-Euler solution of the Appell-Hamel dynamical system with a constant force applied in the tangential direction of the wheel trace. It is found that this solution is consistent with the result of the Chetaev model but at variance with the vakonomic model. We conclude therefore that the vakonomic model does not describe the real motion of nonlinear nonholonomics system.

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