Geometric origin of dynamically induced freezing of quantum evolution

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The phenomenon of dynamical, field-induced freezing of quantum evolution is discussed. It occurs when a time-dependent state is dynamically driven in such a way that the evolution of the corresponding wave function is effectively localized within a small region in the projective Hilbert space. As a consequence, the dynamics of the system is *frozen* and the expectation values of all physical observables hardly change with time. Necessary and sufficient conditions for inducing dynamical freezing are inferred from a general analysis of the geometry of quantum evolution. The relevance of the dynamical freezing for a sustainable in time, dynamical control is discussed and exemplified by a study of the coherent control of the kicked rotor motion.

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Localization phenomena in quantum systems are in the focus of intensive investigations due to their conceptual and technical significance. For example, the so-called Anderson localization (AL) [1], a phenomenon that has been at the heart of the study of transport properties in mesoscopic systems, occurs in disordered solids, and refers to the localization of the charge carrier's wave function in the configuration space. Dynamical localization (DL), on the other hand, implies localization in the phase space [2,3]. Although there are some formal similarities between AL and DL, unlike the former, DL has an intrinsic dynamical origin. The periodically kicked rotor [2,3] constitutes a paradigmatic model for studying DL. In a classical scenario the periodically kicked rotor (e.g., a rotational pendulum periodically kicked by an external force) becomes chaotic after application of a few kicks [2,3] and its motion becomes diffusive. In the quantum regime, however, the classical diffusive motion is suppressed by quantum interference effects that, eventually, lead to localization in the phase space.

In the present Brief Report we discuss a different phenomenon that we term *dynamical freezing* (DF). It refers to a dynamical, field-induced localization of the wave function of the system within a small region of the corresponding projective Hilbert space. Thus, if $\Psi(t_0)$ represents the time-dependent wave function of a system at a given time $t=t_0$, our interest concerns the possibility of *freezing* the subsequent evolution of the system by guaranteeing that $\Psi(t>t_0)$ remains close (up to a phase and for a long enough time) to $\Psi(t_0)$. A geometric quantity that serves as a measure of the angle in Hilbert space between the rays corresponding to $\Psi(t_0)$ and $\Psi(t)$ is the Bargmann angle $\beta(t)$ [4], defined as $(|\cos[\beta(t)]|=|\langle\Psi(t_0)|\Psi(t)\rangle|$. In terms of the Bargmann angle, DF occurs when the system evolves in such a way that $|\cos[\beta(t)]|\approx 1$ for $t\geqslant t_0$. Obviously, such a situation occurs always for stationary systems but this is a trivial case in which we are not interested here. For time-dependent systems, however, such a situation does not occur spontaneously but, as shown here, can be achieved by applying an appropriate time-dependent external field. The purpose of the present report is to study the geometric origin of the DF. From such a study we then determine necessary and sufficient conditions for dynamically freezing a time-dependent quantum state as well as the field parameters capable of inducing DF.

We consider nonrelativistic, nonstationary systems that are well described as a coherent superposition of a finite number N of unperturbed nondegenerate stationary state vectors $|n\rangle$, i.e., the time-dependent state vector is cast as $|\Psi(t)\rangle=\Sigma_{n=1}^{N}c_n(t)|n\rangle$. The evolution of the system is determined by the N-dimensional complex vector $\mathbf{C}(t)\in\mathbb{C}^N$, whose components are the expansion coefficients $c_n(t)$. In the $|n\rangle$ representation the Hamiltonian of the system can be written as $H=\Sigma_{n=1}^{N}E_nP_{nn}+\Sigma_{m,n=1}^{N}\langle m|V(r,t)|n\rangle P_{mn}$, where E_n is the eigenenergy associated with $|n\rangle$, $P_{mn}=|m\rangle\langle n|$, and $V(r,t)$ describes coupling to the time-dependent external field.

For a system with unitary evolution, the corresponding Hamiltonian is Hermitian and can therefore be decomposed as $H=q(t)\mathbb{I}+\mathcal{H}(t)$, where $q(t)=N^{-1}\mathrm{Tr}(H)$, $\mathbb{I}$ is the $N\times N$ unit matrix, and $\mathcal{H}(t)$ belongs to the su(N) algebra. The first summand in the decomposition can be eliminated upon a gauge transformation, hence the physics of the system is governed by $\mathcal{H}(t)$. Since $\mathcal{H}(t)\in\mathrm{su}(N)$, it can be expanded in the $N_D=\dim[\mathrm{su}(N)]$ generators λ_i of su(N), i.e.,

$$\mathcal{H}(t)=\frac{1}{2}\sum_{i=1}^{N_D}a_i(t)\lambda_i. \tag{1}$$

Choosing the generators λ_i such that $\mathrm{Tr}(\lambda_i\lambda_j)=\delta_{ij}$ (here δ_{ij} is the Kronecker symbol), the expansion coefficients in (1) can be cast as $a_i(t)=\mathrm{Tr}[\mathcal{H}(t)\lambda_i]$.

For our purposes it is convenient to make a transformation from the complex Hilbert space $\mathbb{C}^N$ to a real space. This is done by introducing the (Bloch) coherence vector [5–7] $\mathbf{B}(t)$ via the following map Ξ:

$$\mathbf{C}(t)\in\mathbb{C}^N:\Xi[\mathbf{C}(t)]=\mathbf{B}(t)=\mathbf{C}^{\dagger}(t)\mathbf{\Lambda}\mathbf{C}(t)\in\mathbb{R}^{N_D}, \tag{2}$$

where $\mathbf{\Lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_{N_D})$. Thus, $\mathbf{B}(t)$ is a vector whose components are the expectation values of the N_D generators of su(N).

In terms of the coherence vector, the density matrix $\rho(t)$ is determined by [5–7]

$$\rho(t) = \frac{\mathbb{I}}{N} + \frac{1}{2}\sum_{i=1}^{N_D} B_i(t)\lambda_i. \tag{3}$$

Taking into account that $\mathbf{C}(t)$ obeys the time-dependent Schrödinger equation with the Hamiltonian (1) and considering the transformation (2), one obtains the following equation of motion for the coherence vector,

$$\frac{d\mathbf{B}}{dt} = \alpha(t)\mathbf{B}(t), \tag{4}$$

where $\alpha(t)$ is an $N_D \times N_D$ real matrix with elements $\alpha_{kl}(t) = i\mathrm{Tr}([\lambda_k, \lambda_l]\mathcal{H}(t))/(2\hbar)$. The formal solution of (4) can be written as $\mathbf{B}(t) = \upsilon(t,t_0)\mathbf{B}(t_0)$, with $\upsilon(t,t_0) = \hat{T}\exp[\int_{t_0}^{t}\alpha(t')dt']$ and $\hat{T}$ being the time-ordering operator. Since $\upsilon(t,t_0) \in \mathrm{SO}(N_D)$, its action preserves the scalar product and the length of the coherence vector $|\mathbf{B}(t)| = \sqrt{2(N-1)/N}$ is a constant of motion. On the other hand $0 \leqslant \mathrm{Tr}(\rho(t)\rho(t')) \leqslant 1$ (recall we are considering pure states) and one obtains from (3) the following constraint [6,7]:

$$\frac{-2}{N} \leqslant \mathbf{B}(t)\cdot\mathbf{B}(t') \leqslant \frac{2(N-1)}{N}, \quad \forall\ t,t'. \tag{5}$$

Within the coherence vector formalism the DF of the system evolution occurs when a time-dependent external field forces the coherence vector $\mathbf{B}(t)$ to remain in the vicinity of its initial value $\mathbf{B}(t_0)$, i.e., if the relation $\mathbf{B}(t) \approx \mathbf{B}(t_0)$ holds during a long enough time.

One of the geometric quantities that characterizes the quantum evolution of a system is the quantum distance $L(t,t_0) = \int_{t_0}^{t} ds$. Here $ds = 2\sqrt{1 - |\langle\Psi(t)|\Psi(t+dt)\rangle|^2}$ is the infinitesimal distance as measured by the Fubini-Study metric [8–10]. If during the time interval $[t_0,t]$ the state of the system describes a curve in the Hilbert space $\mathbb{C}^N$, then $L(t,t_0)$ is the distance traveled during the interval $[t_0,t]$. To precise this statement we use (2) and (4) to obtain (after some algebra) an expression for the quantum distance in terms of the coherence vector

$$L(t,t_0) = \int_{t_0}^{t} \left|\frac{d\mathbf{B}(t')}{dt'}\right| dt'. \tag{6}$$

The quantum distance coincides thus with the Euclidean length of the curve described by $\mathbf{B}(t)$ during its evolution.

In what follows we limit our analysis to the case of systems with periodic cyclic evolution (PCE) (for a discussion on how to induce periodic cyclic evolution, see Ref. [11], where conditions for determining the field parameters capable of inducing PCE were found). In such a case the time-dependent state vector returns to its initial value (up to a phase) after each evolution cycle, i.e., $|\Psi(t_0+k\mathcal{T})\rangle = \exp[i\phi_k]|\Psi(t_0)\rangle$, where $\mathcal{T}$ is the duration of each periodic cycle, $k = 0,1,2,\ldots,n_{pc}$ (with n_{pc} the number of periodic cycles), and ϕ_k is the total phase acquired during the kth periodic evolution cycle. Note that if the system undergoes a PCE then the coherence vector becomes periodic, $\mathbf{B}(t_0+k\mathcal{T}) = \mathbf{B}(t_0)$ and for each cycle $\mathbf{B}(t)$ describes a closed trajectory ξ_B.

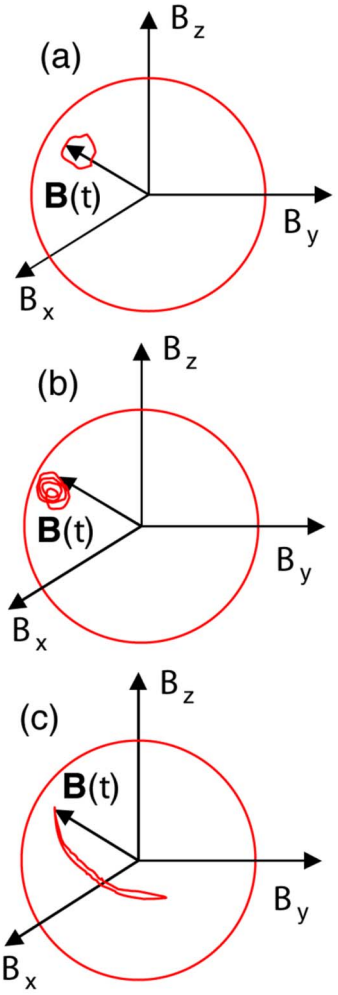

FIG. 1. (Color online) Schematics of the cyclic evolution of the coherence vector $\mathbf{B}(t)$ on the Poncaré sphere. L and $\phi_{AA} = \Omega/2$ [with Ω being the solid angle subtended by the closed trajectory of $\mathbf{B}(t)$ at the center of the sphere] are the quantum distance and the AA phase, respectively. (a) Dynamical freezing with $L \approx 0$, $\phi_{AA} \approx 0$. (b) Dynamical freezing with $L \gg 0$, $\phi_{AA} \approx 0$. (c) $L \gg 0$, $\phi_{AA} \approx 0$ but dynamical freezing does not occur. Comparison of (a) and (b) shows that $L \approx 0$ is sufficient but not necessary. On the other hand, it is clear from (a) and (c) that $\phi_{AA} \approx 0$ is necessary but not sufficient.

For the case $N=2$, i.e., a two-level system (TLS), the evolution of the system in the coherence vector space is characterized by the curve that $\mathbf{B}(t)$ describes on the two-dimensional unit sphere $\mathbb{S}^2$ (also called the Poincaré sphere) embedded in the Euclidean three-dimensional space $\mathbb{R}^3$. We recall that the quantum distance $L(t_0+\mathcal{T},t_0)$ traveled during an evolution cycle is determined by the Euclidean length of the corresponding closed trajectory ξ_B described by $\mathbf{B}(t)$ on the Poincaré sphere [see Eq. (6)]. Hence, a *sufficient* condition for the occurrence of DF is the quantum distance traveled during each periodic evolution cycle to be small [see Fig. 1(a)], i.e., $L_\xi = L(t_0+k\mathcal{T}, t_0+(k-1)\mathcal{T}) \approx 0$ for $k = 1,2,\ldots,n_{pc}$ [11]. Obviously, this condition is not neces-

sary for the DF, since for evolutions corresponding to multiple turns of the coherence vector in the vicinity of $\mathbf{B}(t_0)$ the quantum distance can be rather long while the motion of the coherence vector is still frozen [see Fig. 1(b)]. This sufficient condition for DF is also inferred in a straightforward way for an N-level system.

Another geometric quantity of particular relevance for studying the quantum evolution of cyclic in time systems is the Aharonov-Anandan geometric phase [8–10]. During an evolution cycle the wave function of the system acquires a total phase $\phi=\phi_D+\phi_{AA}$, where ϕ_D and ϕ_{AA} represent the dynamical and the Aharonov-Anandan (AA) phases, respectively. For a given physical system the dynamical phase is only defined up to a gauge transformation and, therefore, is not uniquely determined. On the contrary, the AA phase is gauge invariant and constitutes a uniquely determined property of the physical system, i.e., ϕ_{AA} is a geometric quantity in the sense that it does not depend on the choice of the Hamiltonian but only on the trajectory of the coherence vector. The AA phase acquired during an evolution cycle constitutes a measure of the area subtended by the corresponding closed trajectory ξ_B described by $\mathbf{B}(t)$. For the case of a TLS we have $\phi_{AA}=\Omega/2$ [8], where Ω is the solid angle subtended by ξ_B at the center of the Poincaré sphere. From this analysis one deduces that a *necessary* condition for achieving DF is $\phi_{AA}\approx 0$ [see Figs. 1(a) and 1(b)]. Such a condition, however, is not sufficient [12]. This results clear from the example shown in Fig. 1(c). In such a case, although the condition $\phi_{AA}\approx 0$ is fulfilled, the maximum deviation of $\mathbf{B}(t)$ from its initial value can be significantly large [cf. Fig. 1(c)]. We note that, in general, the solid angle interpretation of ϕ_{AA} breaks down for $N>2$ [13]. Nevertheless, ϕ_{AA} is still a measure of the area enclosed by the closed trajectory described by $\mathbf{B}(t)$ and the above-mentioned necessary condition is also valid for a general N-level system.

As a measure of the maximum deviation of $\mathbf{B}(t)$ from its initial value $\mathbf{B}(t_0)$ during a periodic cycle, we introduce the parameter

$$\varepsilon = \max_{t\in[t_0,t_0+\mathcal{T}]}(|\mathbf{B}(t)-\mathbf{B}(t_0)|). \tag{7}$$

This parameter characterizes the degree of freezing: the smaller the value of ε, the stronger the DF. Then, a *necessary* and *sufficient* condition for DF can be written as $\varepsilon<1$ [note that $0\leqslant\varepsilon\leqslant 2$, as a consequence of the constraint (5)].

Taking into account that the length of the coherence vector is a constant of motion one finds

$$|\mathbf{B}(t)-\mathbf{B}(t_0)| = 2|\mathbf{B}(t)|\sqrt{1-\cos(\gamma(t))}, \tag{8}$$

where the correlation angle $\gamma(t)=\angle(\mathbf{B}(t),\mathbf{B}(t_0))$. From the map (2) one finds that the correlation and Bargmann angles are related through the expression $\cos[\gamma(t)]=(N(\cos[\beta(t)])^2-1)/(N-1)$. Combining this relation with Eqs. (7) and (8), it follows that

$$\varepsilon = \max_{t\in[t_0,t_0+\mathcal{T}]}(2|\sin[\beta(t)]|). \tag{9}$$

The field parameters capable of inducing optimal DF are those for which ε is minimized. Alternatively to ε, one can also use the mean correlation angle $\langle\gamma\rangle_{\mathcal{T}}=\arccos[(1/\mathcal{T})\int_{t_0}^{t_0+\mathcal{T}}\cos[\gamma(t)]dt]$ or $Y(t)=|\cos[\beta(t)]|$ as measures of the degree of DF.

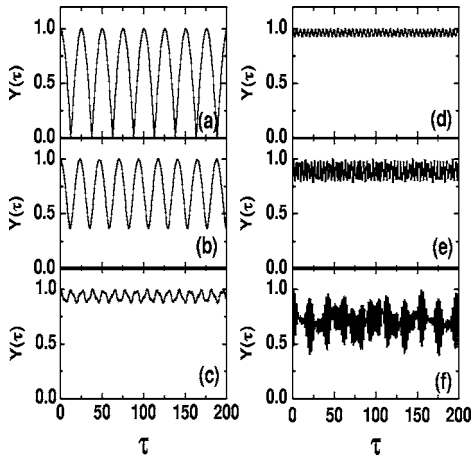

FIG. 2. Periodic cyclic evolution of the cosine of the Bargmann angle $Y(\tau)=|\cos[\beta(\tau)]|$ for a KR with $\kappa=0.5$. (a) $K=0$, $\mathcal{T}/T=25$, $\phi_{AA}=1$, $L=1$, $\varepsilon=2$. (b) $K=0.07$, $\mathcal{T}/T=47$, $\phi_{AA}=0.49$, $L=2.4$, $\varepsilon=1.86$. (c) $K=0.31$, $\mathcal{T}/T=105$, $\phi_{AA}=0.34$, $L=10.6$, $\varepsilon=0.98$. (d) $K=0.59$, $\mathcal{T}/T=63$, $\phi_{AA}=0.17$, $L=9.3$, $\varepsilon=0.74$. (e) $K=1.77$, $\mathcal{T}/T=96$, $\phi_{AA}=0.21$, $L=34.1$, $\varepsilon=1.23$. (f) $K=5.3$, $\mathcal{T}/T=155$, $\phi_{AA}=0.55$, $L=357.5$, $\varepsilon=1.83$.

The main goal of a quantum dynamical control process consists in promoting the system from its initial state to a target state for which at certain time $t=t_0$ the expectation value of an observable O of interest reaches a predefined (target) value $\langle O\rangle(t_0)=\langle O\rangle_t$ that is not accessible within the stationary states of the system. A step further consists in maintaining the values of $\langle O\rangle(t)$ close to its target expectation value $\langle O\rangle_t$ for a desired time interval $[t_0,t']$, i.e., to make the control process *sustainable* in time. This issue is crucial for applications. For example, in most cases chemical reactions are sensitive to the relative orientation of the reactants. The relative orientation (the observable of interest in this case) can be coherently controlled but for the reaction yield the control process has to be sustainable, i.e., the relative orientation has to persists for a time not shorter than the reaction time.

The representation of a given observable O (determined by the operator $\hat{o}$) in the representation of the N stationary levels of the unperturbed system reads $O=\Sigma_{m,n}^{N}O_{mn}P_{mn}$, where $O_{mn}=\langle m|\hat{o}|n\rangle$. Within the coherence vector approach, the expectation value

$$\langle O\rangle(t) = \frac{1}{N}\sum_{n=1}^{N}O_{nn}+\frac{1}{2}\mathbf{W}\cdot\mathbf{B}(\mathbf{t}), \tag{10}$$

where $\mathbf{W}$ is a vector with elements $w_i=\mathrm{Tr}(O\lambda_i)$. Under dynamical freezing conditions, the motion of $\mathbf{B}(t)$ during the time interval $[t_0,t']$ is localized in the vicinity of $\mathbf{B}(t_0)$. Then, as one can easily see from (10), the expectation value of any observable (in fact, of all physical observables) remains close to its corresponding value at $t=t_0$, i.e., the value $\langle O\rangle(t_0)$ is sustained during the time interval $[t_0,t']$. Thus, the

DF is of crucial importance for the coherent sustainability of a control process.

For a numerical illustration of our theoretical predictions we consider the dynamics of a periodically kicked rotor (KR). The quantum KR has experimentally been realized by exposing cold atoms to a pulsed standing wave of light [14]. It has also been considered as a useful model in condensed matter physics [15] and molecular physics [16]. The time-dependent Schrödinger equation of the parametrized, periodically KR reads $i\partial_\tau \Psi = H(\tau)\Psi$, where [3] $H(\tau) = \ell^2 - K\cos\theta \Sigma_{j=0}^{\infty}\delta(\tau - j)$, τ is the time in units of the period T of the kicks, $\ell = i\kappa\partial_\theta$, $\kappa = \sqrt{\hbar T/(2I)}$ (with I being the moment of inertia), and $K = V_0 T/\hbar$ (V_0 is the strength of the kicking field). We consider that the rotor is initially (before the application of the kicks) in a time-dependent state $|\Psi(t)\rangle = \Sigma_n c_n(t \leqslant 0)|n\rangle$ (with $|n\rangle$ denoting the states of the rotor in absence of the field) such that $c_0(0) = 1/\sqrt{2}$, $c_{\pm 1}(0) = 1/2$, and $c_n(0) = 0$ for $|n| \geqslant 2$ [17]. The evolution operator over a period of the kicking field is given by $U(\tau = 1, 0) = e^{-i\ell^2} e^{iK\cos\theta}$. We fix the value of T and determine the kick strengths (i.e., the values of K) for which the system undergoes a cyclic evolution (for details on how to find the field parameters leading to the PCE of a time-dependent system, see Ref. [11]). We then calculate the quantum distance traveled during an evolution cycle by integrating the Fubini-Study metric from $\tau = 0$ to $\tau = \mathcal{T}/T$. The corresponding AA geometric phase is obtained as $\phi_{AA} = \phi - \phi_D$, where $\phi = \langle\Psi(0)|\Psi(\mathcal{T}/T)\rangle$ and $\phi_D = (-1/\hbar)\int_0^{\mathcal{T}/T}\langle\Psi(\tau)|H(\tau)|\Psi(\tau)\rangle d\tau$ are the total and dynamical phases, respectively. In Fig. 2 we show the time dependence of the cosine of the Bargmann angle $Y(\tau) = |\cos[\beta(\tau)]|$ for some values of K for which the system undergoes PCE. The values $\mathcal{T}$ (the duration of a periodic cycle) are also given for each case. The corresponding Aharonov-Anandan geometric phase ϕ_{AA} (in units of π), the quantum distance L (in units of 2π), and the values ε of the degree of DF were computed over a periodic cycle and are specified in the figure caption. Under field-free evolution [see Fig. 2(a)], $\phi_{AA} = 1$ (recall phases are given in units of π), i.e., the necessary condition for DF is not fulfilled and the system oscillates between orthogonal states with the rotational period of the rotor $\mathcal{T}/T = 2\pi/\kappa^2 \approx 25$. While ϕ_{AA} is relatively small in Figs. 2(c)–2(e), the sufficiency condition $L \approx 0$ does not occur [18]. In Figs. 2(a), 2(b), and 2(f) DF is not achieved since the necessary and sufficient condition $\varepsilon < 1$ is not fulfilled. However, as the value of ε decreases the DF becomes stronger [compare Figs. 2(c)–2(e)] and for $\varepsilon = 0.74$ [see Fig. 2(d)] the time evolution of the system is optimally frozen and the deviation of $Y(\tau)$ from 1 is less than 1% at any time. In such a situation the main value of any observable varies very little in time. In fact, we calculated the main values of various observables (e.g., $\langle\ell\rangle$, $\langle\ell^2\rangle$, $\langle\cos\theta\rangle$, $\langle\theta\rangle$) for the case shown in Fig. 2(d) and we found (not shown here) that, indeed, during the system's time evolution the main values of these observables remain very close to their respective initial values.

In summary, the phenomenon of dynamical freezing was studied. Necessary and sufficient conditions for inducing dynamical freezing were deduced from the geometrical analysis of the quantum evolution. Numerical evidence of the dynamical freezing of a kicked quantum rotor was presented.

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[13] For $N > 2$ the motion of the point determined by $\mathbf{B}(t)$ does not cover all the hypersphere $\mathbb{S}^{N_D-1}$ of radius $|\mathbf{B}(t)|$ but is restricted to a submanifold $\mathbb{D}$ of $\mathbb{S}^{N_D-1}$ [7]. Consequently, codimensional problems appear and the solid angle picture breaks down.

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[18] In Figs. 2(c)–2(e) the dynamical freezing is of the type shown in Fig. 1(b) with $L \gg 0$ and small ϕ_{AA}.